around the earth's curvature by the tenuous atmosphere (see Ref. 9). Ivanov attributes these effects to a shock wave generated by the explosion impinging on the ionosphere although he recognizes some inadequacies in this explanation. He appears to overlook one important fact which we pointed out. The directions of the magnetic disturbance vectors at significant stations are compatible with a current sheet extending more than 2000 km from the explosion point, but are incompatible with magnetic effects due to currents limited to distances less than 1250 km from the origin, such as would have been produced by the shock wave.

For this reason the reviewer believes that the shock wave hypothesis is an inadequate explanation for the effects associated with these events, although it may suffice for explanation of that of the Christmas Island event and the Tunguska meteorite effect.

It is not unlikely that several mechanisms were involved, just as it is apparent that the effects at Apia were due to charged particles traveling along the lines of the earth's magnetic field. Perhaps when the effects of more recent explosions have been carefully studied our understanding of the phenomena will be improved.

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Combination of the Monte Carlo Method with the Method of Steepest Descents for the Solution of Certain Extremal Problems

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In the report[1] at the All-Union Conference on Computational Mathematics and Computational Technology (Moscow, November 1959) two methods were presented which permitted a significant reduction in the number of runs required in finding extreme values of a function of many variables by the Monte Carlo method. These were a combination of the Monte Carlo method with the classical methods of analysis and a combination of the Monte Carlo method with linear programming. Both methods were illustrated with concrete examples.

In the present note a third method will be illustrated with a concrete problem: a combination of the Monte Carlo method with the method of steepest descents.

1. Statement of the Problem

Let there be given in the plane $n \geqq 3$ points with coordinates (x_i,y_i) $(i = 1,2,\ldots,n)$, not all lying on a single straight line, and n positive numbers k_i. It is required to find m centers $(m < n)$ with coordinates (a_j,b_j) $(j = 1,2,\ldots,m)$ which will minimize the function

$$f(a_1,b_1,a_2,b_2,\ldots,a_m,b_m) = \sum_{i=1}^{n} k_i \sqrt{(x_i - a_{j(i)})^2 + (y_i - b_{j(i)})^2} \quad (1)$$

which is constructed as follows.

Each point (x_i,y_i) is attached to the nearest of the points (a_j,b_j), which is denoted in the foregoing by $(a_{j(i)},b_{j(i)})$. If the distance from the point (x_i,y_i) takes on its minimum at several (say p) of the points (a_j,b_j), it is then attached to each of these points, and enters the sum [Eq. (1)] p times with coefficient k_i/p. In the case $m = 1$ the problem reduces to the well-known Steiner problem, the solution of which we shall briefly discuss.

2. Solution of the Problem in the Case $m = 1$

In this case it is easy to see that the function $f(a,b)$ is convex, and that there exists a unique solution of the problem, the Steiner point (a_0,b_0). Here the solution may be general or singular. We call a solution singular if the point (a_0,b_0) coincides with one of the original points (x_i,y_i).

In case the solution is general, it satisfies the conditions

$$\frac{\partial f(a,b)}{\partial a} = \frac{\partial f(a,b)}{\partial b} = 0 \quad (2)$$

If the solution is singular, for example $a_0 = x_{i_0}$, $b_0 = y_{i_0}$, we then have

$$k_{i_0} - \sqrt{\left(\frac{\partial \tilde{f}_{i_0}}{\partial a}\right)^2 + \left(\frac{\partial \tilde{f}_{i_0}}{\partial b}\right)^2} \geqq 0 \quad (3)$$

where

$$\tilde{f}_{i_0} = f(a,b) - k_{i_0} \sqrt{(a - x_{i_0})^2 + (b - y_{i_0})^2} \quad (4)$$

To solve the problem in the case $m = 1$ we use the following algorithm, which is based on the method of steepest descents. As an initial point we take the center of gravity of the given system of points. We are given an initial step Δ_0, and move in the direction opposite to the gradient of $f(a,b)$ at the initial point.

We move from the point a step Δ_0 in the direction opposite to the gradient. If at the point thus reached, $r_i \leqq \epsilon$ (where r_i is the distance from the point (x_i,y_i) and ϵ is a sufficiently small positive number), we check the point (x_i,y_i) to see whether condition (3) is satisfied. If (3) is satisfied, the problem is solved. Otherwise, we take the point (x_i,y_i) as initial point. If there is no i such that $r_i \leqq \epsilon$, we check to see whether the condition

$$|\operatorname{grad} f(a,b)| < \epsilon \quad (5)$$

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is satisfied. If Eq. (5) holds, the problem is solved. Otherwise we determine the direction opposite to the gradient at the new point. Let it make an angle $\Delta\gamma$ with the direction stored in the memory. Then

a) if

$$\Delta\gamma < 45° + \delta \tag{6}$$

where δ is a sufficiently small number, the motion proceeds with the same step length in the old direction;

b) if

$$\Delta\gamma > 90° - \delta \tag{6'}$$

we divide the step by two, and go in the opposite direction;

c) If neither of conditions (6) and (6') is fulfilled, we move in the new direction opposite to the gradient.

It can be shown that in a finite number of steps the process terminates, with either (3) or (5) fulfilled. This gives the solution to the problem. The condition that each time the direction changes it is through an angle

$$\Delta\gamma \geqq 45° + \delta$$

insures the following property of the descent trajectory: a point (x_i,y_i) which is not the solution of the problem cannot be approximated arbitrarily closely. This reduces the time of solution of the problem.

The algorithm just described has been programmed for machine use, and a number of problems have been solved using it. If the maximal distance between points (x_i,y_i) is taken as a unit, it is recommended that $\Delta_0 = 0.1$ and $\epsilon = 0.0001$. A smaller step only increases the number of iterations, and similarly a decrease in ϵ increases the number of iterations and increases the accuracy only slightly. Already for $\epsilon = 0.0001$, as is shown by comparison with $\epsilon = 0.000001$, the error in determining the center does not exceed 0.0001, and the error in determining min $f(a,b)$ does not exceed 0.000001.

The solution time of the problem for 10–20 points with such accuracies is roughly 1–2 sec.

3. Solution of the Problem in the General Case

In the general case $m > 1$, we solve the problem by the Monte Carlo method in combination with the method set forth above. We take m randomly placed centers as initial positions and improve them by a series of iterations. Each iteration consists of two steps: 1) attaching the points (x_i,y_i) to the centers obtained at the previous iteration, and 2) shifting each center by the method set forth in Sec. 2, so as to minimize the weighted sum of its distances from the attached points.

As is shown by experiment, after several iterations (the number of iterations is as large as 9 only in very exceptional cases) the centers become fixed. As for the sum of distances obtained in the process, it must converge since it decreases at each iteration.

However, in contrast to the case $m = 1$, the general problem has a number of local minima. Therefore it is necessary to take many initial arrangements of the centers, and from among the improved arrangements obtained by iteration to choose those which give the least values of the function. The process is continued until the least value of the function is repeated several times, and the more times it is repeated, the greater is the probability that we have actually obtained the least value.

We shall give the results of the solution for three examples. Each example was run for 10–11 min by the method set forth, and for roughly the same time by the Monte Carlo method without steepest descents. Analysis of the results obtained leads to some qualitative conclusions.

Example 1: $n = 10$, $m = 3$. The combined method was run 84 times, and 14 local minima were discovered. The values obtained were:

$b_{\min}$	No. of times
68.34	8
69.14	7
72.11	17

The remaining local minima gave values at most 5% in excess of $f_{\min}$. Each minimum corresponded to a definite position for the centers.

The usual Monte Carlo method made 4116 runs in the same period of time. Here there was achieved

$$f_{\min}' = 75.25$$

which gives an error of roughly 10%.

The further values obtained were distributed as follows:

f	No. of times
75–80	23
80–85	79
85–90	130

etc. Thus if an accuracy of better than 10% is required, it can be obtained in the time indicated only by the combination of methods. If we are allowed an error of the order of 17%, then the combination of methods (77 runs out of 84) is better than the usual method (23 runs out of 4116). If an error of the order of 25% is allowed, the combination of methods is still better (83 runs out of 84 as against 102 out of 4116).

Example 2. $n = 30$, $m = 3$ (points were distributed in a nonuniform manner throughout a region.) In 10 minutes 40 runs were made with the combination of methods, with the results:

$f_{\min}$	No. of times
316.3	27
331.8	5
335.8	1
345.6	7

With the usual Monte Carlo method, 868 runs were made in 6 min with the result:

$$f_{\min}' = 353.2$$

which gives an error of the order of 11%.

Example 3. $n = 50$, $m = 3$ (points distributed relatively uniformly). In 10 min 31 runs were made with the combination of methods with results:

$f_{\min}$	No. of times
660.7–660.98	7
661.5–662	2
664.1–665.96	16
670.9	4
690 and above	2

With the usual method 1113 runs (in roughly 10 min) gave:

$f_{\min}' = 677.7$	
$f \quad < 690$	5 times

In this case, because of the relatively uniform distribution of the points, the minima were less sharply expressed, and the error of the usual method was 2.6%. But if the time of calculation is reduced by a factor of 10, the error with the combination of methods does not exceed 1%, whereas the error with the usual method exceeds 5%.

Thus in this case also the combination of methods is certainly to be preferred.

We note that the time necessary is almost independent of the number of points n. For large numbers of points we may make a preliminary grouping, considering several closely distributed points as one, located at the center of gravity

of the group, and summing the corresponding coefficients k_i. Application of such a method to example 3 showed that the accuracy of the solution remained satisfactory, while the time necessary was reduced insignificantly.

—*Received February 2, 1961*

Reference

[1] Gurin, L. S., "An experiment applying the Monte Carlo method for the location of extreme values of a function," *Computational Mathematics and Computational Technology* (Mašgiz, Moscow, 1962).

Reviewer's Comment

The problem discussed here has been relegated mostly to mathematical recreations in the western literature, although it has appeared in serious (oral) discussions as a "warehouse location" problem: given n points in the plane, find m points—"centers" or "distribution centers"— such that, if each of the points is joined to some center by a line segment, the sum of the lengths of the segments is minimized. An account of the problem, due to Steiner, for $m = 1$ is given in *What Is Mathematics* by Courant and Robbins (Oxford University Press, New York, 1941), in which it is remarked that the general problem does not lead to interesting (theoretical) results. However, this problem remains useful in applications.

The problem is a discrete-nonlinear programming problem and could, in principle, be solved exactly by some recent extensions of current methods for discrete programming problems. The discrete feature is handled here, however, in what is probably the best practical way, by the "Monte Carlo" device of selecting a group of centers at random and assigning each point to the nearest center. The nonlinear portion is handled by an ordinary gradient method, although the authors' device of insuring that the gradient direction always changes by at least 45° so that "a point which is not the solution of the problem cannot be approximated arbitrarily closely" is new. Since, however, the function being minimized is convex, it does not seem that that difficulty could arise here, but the device is interesting in general. It is unfortunate that evidence for a reduction of solution time is not cited, and that the computer used is not described so that the meaning of the cited solution times can be understood.

The problem of finding a single center has a neat mechanical analogue. It is the equilibrium position of a point acted on by n unit forces, each of magnitude k_i directed toward the ith given point. A system of string, weights, and pulleys embodying this principle would probably be an effective alternative means of solving this problem.

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Digest of Translated Russian Literature

The following abstracts have been selected by the Editor from translated Russian journals supplied by the indicated societies and organizations, whose cooperation is gratefully acknowledged. Information concerning subscriptions to the publications may be obtained from these societies and organizations. Note: Volumes and numbers given are those of the English translations, not of the original Russian.

APPLIED MATHEMATICS AND MECHANICS (Prikladnaia Matematika i mekhanika). Published by American Society of Mechanical Engineers in conjunction with Pergamon Institute

Volume 26, Number 2, 1962

Nonsteady Propagation of Cracks, G. I. Barenblatt, R. L. Salganik, and G. P. Cherepanov, pp. 469–477.

Investigations of the processes of crack propagation have continued now for a considerable period of time, and it would be fair to say that in the field of stationary propagation of cracks the investigations have more or less reached completion.

One of the simplest problems of nonstationary propagation of cracks would appear to be the problem of the widening at constant velocity of a rectilinear crack in a uniform stress field perpendicular to the line of the crack. This problem has been investigated by a number of authors, starting with Mott, but it was not until the paper by Broberg that it was treated as a problem of the dynamic theory of elasticity. Broberg, however, neglected the effect of cohesive forces, and for this reason came to the conclusion that the uniform propagation of cracks can take place only at a velocity equal to the velocity of propagation of Rayleigh surface waves: at any other velocity an uncompensated singularity occurs in the stress field at the end of the crack.

The present paper investigates on the basis of certain assumptions the effect of cohesive forces and derives an equation which defines the velocity of propagation of a crack in terms of the applied stress. It is shown that for every material there is a certain minimum velocity of stable uniform crack propagation. It is also shown that the velocity of stable propagation of a crack increases with increase in the splitting force and tends to the Rayleigh velocity: it would appear that in isotropic bodies the formation of a regime of uniform propagation at the Rayleigh velocity is prevented by the occurrence of branching of the crack.

Formulation of Refined Theories of Plates and Shells, I. G. Teregulov, pp. 495–502.

The attempt to refine theories of plates and shells was started in other studies, and at the present time many papers are devoted to this problem. These papers usually use one of a number of assumptions. A survey of them is beyond the scope of this note. We mention only the papers in which a certain error is specified at the outset, for example, of the order of h^4/L^4 compared to unity ($2h$ is the thickness and L is the transverse dimension of the plate) and the differential equations corresponding to this accuracy are obtained. Boundary conditions to within this